

SIGN CHANGES OF FOURIER COEFFICIENTS OF MODULAR FORMS OF HALF INTEGRAL WEIGHT, 2

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ABSTRACT. In this paper, we investigate the sign changes of Fourier coefficients of half-integral weight Hecke eigenforms and give two quantitative results on the number of sign changes.

1. INTRODUCTION

The study of sign-changes of Fourier coefficients of automorphic forms is recently very active. For modular (Hecke eigen-)forms of integral weight, the consequential result from Matomäki and Radziwiłł [14] is exceptionally charming, where the multiplicative properties of the Fourier coefficients play a substantial role. However the modular forms of half-integral weight do not share the same kind of multiplicativity, and many problems deserve delving.

Let $\ell \geq 2$ be a positive integer, and denote by $\mathfrak{S}_{\ell+1/2}$ the set of all cusp forms of weight $\ell + 1/2$ for the congruence subgroup $\Gamma_0(4)$. Consider the coefficients in the Fourier expansion of a complete Hecke eigenform $\mathfrak{f} \in \mathfrak{S}_{\ell+1/2}$ at ∞ ,

$$(1.1) \quad \mathfrak{f}(z) = \sum_{n \geq 1} \lambda_{\mathfrak{f}}(n) n^{\ell/2-1/4} e(nz) \quad (z \in \mathcal{H}),$$

where $e(z) = e^{2\pi iz}$ and \mathcal{H} is the Poincaré upper half plane. A specific question is the number of sign-changes when all $\lambda_{\mathfrak{f}}(n)$ are real. We interlude with the meaning of sign-changes of a sequence.

Let \mathcal{N} be a subset of \mathbb{N} endowed with the ordering of integers. The sets of squarefree integers or arithmetic progressions are basic examples. Given a real sequence $\{a_n\}_{n \in \mathcal{N}}$. A sign-change is realized via a closed and bounded interval $[i, j] \subset (0, \infty)$ such that

- (i) its end-points i, j lie in \mathcal{N} and satisfy $a_i a_j < 0$, and
- (ii) $a_n = 0$ for all $n \in (i, j) \cap \mathcal{N}$.

The sequence $\{a_n\}_{n \in \mathcal{N}}$ is said to have a sign-change in the interval I if I contains one such interval $[i, j]$. Besides, the number of sign-changes of $\{a_n\}_{n \in \mathcal{N}}$ in $[1, x]$, denoted by $\mathcal{C}^{\mathcal{N}}(x)$, is meant to be the number of intervals $[i, j]$ contained in $[1, x]$.[†]

Let \mathfrak{b} be the set of squarefree numbers. Hulse, Kiral, Kuan & Lim [6] proved that the sequence $\{\lambda_{\mathfrak{f}}(t)\}_{t \in \mathfrak{b}}$ has an infinity of sign-changes. A quantitative version is given in Lau, Royer & Wu [13, Theorem 4], which says $\mathcal{C}_{\mathfrak{f}}^{\mathfrak{b}}(x) \gg x^{(1-4\varrho)/5-\varepsilon}$ where $\mathcal{C}_{\mathfrak{f}}^{\mathfrak{b}}(x)$ denotes the number of sign-changes of $\{\lambda_{\mathfrak{f}}(t)\}_{t \in \mathfrak{b}}$ in $[1, x]$ and the constant ϱ is determined by (3.5) below. Conjecturally $\varrho = \varepsilon$ but it is still hard to guess the tight lower bound.

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[†]An equivalent but slightly different formulation is given in [13].

On the other hand, Meher & Murty [15] studied the sign-change problem for Hecke eigenforms \mathfrak{f} in Kohnen plus subspace of $\mathfrak{S}_{\ell+1/2}$. A form \mathfrak{f} in the plus space has its Fourier coefficients supported at integers $n \equiv 0$ or $(-1)^\ell \pmod{4}$, i.e. \mathfrak{f} has the Fourier expansion at ∞ of the form

$$\mathfrak{f}(z) = \sum_{(-1)^\ell n \equiv 0, 1 \pmod{4}} \lambda_{\mathfrak{f}}(n) n^{\ell/2-1/4} e^{2\pi i n z}.$$

When \mathfrak{f} is a Hecke eigenform in the plus space and its coefficients $\lambda_{\mathfrak{f}}(n)$ are all real, Meher & Murty proved in [15, Theorem 2] that $\{\lambda_{\mathfrak{f}}(n)\}_{n \in \mathbb{N}}$ has a sign-change in the short interval $(x, x + x^{43/70+\varepsilon}]$ for any $\varepsilon > 0$ and for all sufficiently large $x \geq x_0(\varepsilon)$. An immediate consequence is $\mathcal{C}_{\mathfrak{f}}^{\mathbb{N}}(x) \gg x^{27/70-\varepsilon}$. This work naturally motivates the sign-change problem for arithmetic progressions.

In this paper, we furnish progress, based on our work in [10], in the above problems for complete Hecke eigenforms $\mathfrak{f} \in \mathfrak{S}_{\ell+1/2}$. Firstly for the case $\mathcal{N} = \mathfrak{b}$, we sharpen the lower bound for $\mathcal{C}_{\mathfrak{f}}^{\mathfrak{b}}(x)$.

Theorem 1. *Let $\ell \geq 2$ be an integer and $\mathfrak{f} \in \mathfrak{S}_{\ell+1/2}$ a complete Hecke eigenform such that its Fourier coefficients are real. Let ϱ be defined as in (3.5) below, and ϑ any number satisfying*

$$0 < \vartheta < \min\left(\frac{1-2\varrho}{3}, \frac{1}{4}\right).$$

Then

$$(1.2) \quad \mathcal{C}_{\mathfrak{f}}^{\mathfrak{b}}(x) \gg_{\mathfrak{f}, \vartheta} x^{\vartheta}$$

for all $x \geq x_0(\mathfrak{f}, \vartheta)$, where the constant $x_0(\mathfrak{f}, \vartheta)$ and the implied constant depend on \mathfrak{f} and ϑ only.

Remark 1. In particular, Conrey & Iwaniec [2] gives $\varrho = \frac{1}{6} + \varepsilon$ which leads to

$$\mathcal{C}_{\mathfrak{f}}^{\mathfrak{b}}(x) \gg_{\mathfrak{f}, \varepsilon} x^{2/9-\varepsilon}$$

for all $x \geq x_0(\mathfrak{f}, \varepsilon)$, improving the exponent $\frac{1}{15} - \varepsilon$ in [13].

Secondly we generalize the case of $\mathcal{N} = \mathbb{N}$ in Meher & Murty [15] to arithmetic progressions. Let $Q \geq 1$ be an integer, and $a = 0$ or $a \in \mathbb{N}$ with $(a, Q) = 1$. Define

$$(1.3) \quad \mathcal{A} = \mathcal{A}_{a, Q} := \{n \in \mathbb{N} : n \equiv a \pmod{Q}\}.$$

We study the sign-changes of $\{\lambda_{\mathfrak{f}}(n)\}_{n \in \mathcal{A}}$ and sharpen the exponent $\frac{43}{70} + \varepsilon$ of Meher & Murty's result to $\frac{1}{2}$, which in turn gives the better lower bound $\mathcal{C}_{\mathfrak{f}}^{\mathbb{N}}(x) \gg x^{1/2}$.

Theorem 2. *Assume the same conditions for \mathfrak{f} and ϱ in Theorem 1. Let $Q \geq 1$ be odd and $\mathcal{A} = \mathcal{A}_{a, Q}$ defined as in (1.3). Suppose one of the following condition holds:*

- 1° $Q = 1$;
- 2° $a = 0$ and $Q = \prod_{p|Q} p^{\alpha_p}$ where all α_p are odd;
- 3° $(a, Q) = 1$ and $Q = \prod_{p|Q} p^{\alpha_p}$ where all α_p are ≥ 2 .

Then there are positive constants $c_0 = c_0(\mathfrak{f}, Q)$ and $x_0 = x_0(\mathfrak{f}, Q)$ such that the sequence $\{\lambda_{\mathfrak{f}}(n)\}_{n \in \mathcal{A}}$ has at least one sign change in the interval $(x, x + c_0 x^{1/2}]$ for all $x \geq x_0$. In particular, we have

$$\mathcal{C}_{\mathfrak{f}}^{\mathcal{A}}(x) \gg_{\mathfrak{f}, Q} x^{1/2}$$

for all $x \geq x_0$.

2. METHODOLOGIES

Let $\lambda_f(n)$ be the coefficients as in (1.1) and \mathcal{N} a subset of \mathbb{N} . Define

$$(2.1) \quad S_f^{\mathcal{N}}(x) := \sum_{\substack{n \leq x \\ n \in \mathcal{N}}} \lambda_f(n).$$

A typical approach for the sign-change detection exploits the oscillation exhibited in the mean $S_f^{\mathcal{N}}(x)$, while to locate the sign-change, the mean over short intervals, i.e. $S_f^{\mathcal{N}}(x+h) - S_f^{\mathcal{N}}(x)$ for small h , will be a good device. Suppose a sign-change is found in the interval $[x, x+h]$ for every x large enough. Then it follows immediately that the number of sign-changes in $[1, x]$ is at least $x/h + O(1)$ (and hence $\gg x/h$). A standard way to study $S_f^{\mathcal{N}}(x)$ is via the Dirichlet series. But for various \mathcal{N} , we get different degree of its analytic information.

For $\mathcal{N} = \mathfrak{b}$, i.e. the case of squarefree integers, we only get an analytic continuation of the Dirichlet series

$$(2.2) \quad L_f^{\mathfrak{b}}(s) := \sum_{t \geq 1}^{\mathfrak{b}} \lambda_f(t) t^{-s}$$

in the half-plane $\Re s > \frac{1}{2}$, where $\sum_{t \geq 1}^{\mathfrak{b}}$ ranges over squarefree integers $t \geq 1$. As illustrated in [13], it turns out that the weighted mean is more effective. Thus, to prove Theorem 1, we first derive (2.3) below,

$$(2.3) \quad \sum_{x \leq t \leq x+h}^{\mathfrak{b}} \lambda_f(t) \min \left\{ \log \left(\frac{x+h}{t} \right), \log \left(\frac{x}{t} \right) \right\} \ll_{\varepsilon} h^{\frac{1}{2}} x^{\varepsilon}.$$

The better exponent $\frac{1}{2}$ (versus $\frac{3}{4}$ in [13]) of h is a key for the improvement. Another key is to have a mean square formula with better O -term. In [13], we showed that

$$\sum_{X < n \leq 2X} |\lambda_f(n)|^2 = D_f X + O_{f,\varepsilon}(X^{\beta+\varepsilon}).$$

with $\beta = \frac{3}{4} + \varrho$. Here we sharpen it to $\beta = \frac{3}{4}$ in Lemma 4.1 and then conclude Theorem 1 with argument in [13]. This will be done in Section 4.

Next for $\mathcal{N} = \mathcal{A}$ (see (1.3)), we shall provide a truncated Voronoi formula for $S_f^{\mathcal{A}}(x)$ in Section 6. This result is itself interesting since the Voronoi formula is an vital tool for many applications, see [7], [11] for example. Then we complete the proof of Theorem 2 with the method of Heath-Brown and Tsang [5]. However the congruence condition underlying \mathcal{A} gives rise to new (but interesting) difficulties. To transform the congruence, additive characters of modulus $d|Q$ will be invoked and then two consequences follow: the summands in the Voronoi formula are intertwined with Kloosterman-Salié sums, and the frequencies in the cosines are of the form \sqrt{n}/d . We need to select a suitable frequency for amplification with a pair of non-vanishing Salié sum and Fourier coefficient in the associated summand. The implementation is successful when Q fulfills the conditions in Theorem 2, which will be elucidated in Sections 7 & 8. It is worthwhile to remark that the mean square result of $\lambda_f(n)$ is not needed for the method in [5].

3. BACKGROUND

A cusp form $\mathfrak{f} \in \mathfrak{S}_{\ell+1/2}$ has Fourier expansions at the three inequivalent cusps $\infty, -\frac{1}{2}, 0$ of $\Gamma_0(4)$, which are respectively given by (1.1), and (3.1), (3.2) below:

$$(3.1) \quad \begin{aligned} \mathfrak{g}(z) &:= 2^{\ell+1/2}(-8z+1)^{-(\ell+1/2)}\mathfrak{f}\left(\frac{4z}{-8z+1}\right) \\ &= 2^{\ell+1/2} \sum_{n \geq 1} \lambda_{\mathfrak{g}}(n) n^{\ell/2-1/4} \mathfrak{e}(nz) \end{aligned}$$

and

$$(3.2) \quad \mathfrak{h}(z) := (-i2z)^{-(\ell+1/2)}\mathfrak{f}\left(\frac{-1}{4z}\right) = \sum_{n \geq 1} \lambda_{\mathfrak{h}}(n) n^{\ell/2-1/4} \mathfrak{e}(nz).$$

Following the argument in [13, Section 2.2], we have

$$(3.3) \quad \sum_{n \leq x} |\lambda_f(n)|^2 \sim x \quad (\text{for all three cases } f = \mathfrak{f}, \mathfrak{g}, \mathfrak{h}).$$

When \mathfrak{f} is a complete Hecke eigenform, we know from [10] that \mathfrak{g} and \mathfrak{h} are Hecke eigenforms of $\mathbb{T}(p^2)$ for all odd prime p . A consequence is, cf. [10, Lemma 3.2 with $\mathcal{Q} = \{2\}$]: for all odd $m \geq 1$, all squarefree t and $j \geq 0$,

$$(3.4) \quad \lambda_f(2^j t) = 0 \Rightarrow \lambda_f(2^j t m^2) = 0 \quad (f = \mathfrak{f}, \mathfrak{g}, \mathfrak{h}).$$

In addition, we have the following pointwise estimate, see [10, Lemma 3.3].

Lemma 3.1. *Let \mathfrak{f} be a complete Hecke eigenform, \mathfrak{g} and \mathfrak{h} be defined as above. For any integer $m = tr^2$ where $t \geq 1$ is squarefree, we have*

$$\lambda_f(m) \ll_{\mathfrak{f}} |\lambda_f(t)| \tau(r)^2 + |\lambda_{\mathfrak{f}}(t)| \tau(r)^2 \ll_{\mathfrak{f}, \varrho} t^{\varrho} \tau(r)^2$$

for $f = \mathfrak{f}, \mathfrak{g}, \mathfrak{h}$ respectively, where $\tau(n)$ is the divisor function and ϱ satisfies (3.5) below. The first implied \ll -constant depends only \mathfrak{f} and the second implied \ll -constant depends at most on \mathfrak{f} and ϱ .

Here ϱ denotes the exponent for which

$$(3.5) \quad \lambda_{\mathfrak{f}}(t) \ll_{\varrho} t^{\varrho} \quad \forall t \text{ squarefree,}$$

i.e. the bound towards the Ramanujan Conjecture for the half-integral weight Hecke eigenforms. The conjectural value is $\varrho = \varepsilon$. Conrey & Iwaniec [2] obtained $\varrho = \frac{1}{6} + \varepsilon$.

Let $d \geq 1$ be an integer and $(u, d) = 1$. Define the twisted L -function for \mathfrak{f} by

$$(3.6) \quad L_{\mathfrak{f}}(s, u/d) = \sum_{m \geq 1} \frac{\lambda_{\mathfrak{f}}(m) \mathfrak{e}(mu/d)}{m^s} \quad (\Re s > 1)$$

and define similarly for \mathfrak{g} and \mathfrak{h} . These twisted L -functions when attached with suitable factors may be expressed as integrals of \mathfrak{f} along vertical geodesics, and extend to entire functions, cf. [6, (4.4)-(4.5)]. Moreover Hulse et al found the functional equation for $L_{\mathfrak{f}}(s, u/d)$, which is put in the following form

$$(3.7) \quad q_d^s L_{\infty}(s) L_{\mathfrak{f}}(s, u/d) = i^{-(\ell+1/2)} q_d^{1-s} L_{\infty}(1-s) \tilde{L}_{\mathfrak{f}}(1-s, v/d),$$

where $uv \equiv 1 \pmod{d}$ and $L_\infty(s) := (2\pi)^{-s} \Gamma(s + \frac{\ell}{2} - \frac{1}{4})$ is the gamma factor, cf. [6, Lemma 4.3] and [10]. The conductor q_d and the dual L -function $\tilde{L}_f(s, v/d)$ are defined as follows:

$$(3.8) \quad q_d = d \text{ or } 2d \text{ according to } 4 \mid d \text{ or not,}$$

and

$$(3.9) \quad \tilde{L}_f(s, v/d) := \sum_{n \geq 1} \lambda(n; d) \varpi_d(n, v) n^{-s},$$

where

$$(3.10) \quad \begin{array}{|c|c|c|} \hline & \lambda(n; d) & \varpi_d(n, v) \\ \hline 4 \mid d & \lambda_f(n) & \varepsilon_v^{2\ell+1} \left(\frac{d}{v}\right) e\left(\frac{-nv}{d}\right) \\ \hline 2 \parallel d & \lambda_g(n) & \varepsilon_v^{2\ell+1} \left(\frac{d}{v}\right) e\left(\frac{-nv}{4d}\right) \\ \hline 2 \nmid d & \lambda_h(n) & i^{\ell+1/2} \varepsilon_d^{-(2\ell+1)} \left(\frac{v}{d}\right) e\left(\frac{-4nv}{d}\right) \\ \hline \end{array}$$

with $4\bar{4} \equiv 1 \pmod{d}$.

In [6], Hulse et al applied $L_f(s, u/d)$ to obtain the analytic properties of $L_f^b(s)$, which was sharpened to the following result [10, Theorem 1].

Lemma 3.2. *For a complete Hecke eigenform $f \in \mathfrak{S}_{\ell+1/2}$, the series $L_f^b(s)$ extends analytically to a holomorphic function on $\Re s > \frac{1}{2}$, and for any $\varepsilon > 0$,*

$$(3.11) \quad L_f^b(s) \ll_{f, \varepsilon} (|\tau| + 1)^{1-\sigma+2\varepsilon} \quad \left(\frac{1}{2} + \varepsilon \leq \sigma \leq 1 + \varepsilon, \tau \in \mathbb{R}\right),$$

where the implied constant depends on f and ε only.

Remark 2. Using Lemma 3.2 in place of [13, Proposition 7], the estimate in (2.3) follows plainly from the same argument as in [13, Section 4.1], so we do not repeat here.

4. PROOF OF THEOREM 1

We start with the following lemma where the O -term in (4.1) is smaller than [13, (14)].

Lemma 4.1. *Let $\ell \geq 2$ be a positive integer and $f \in \mathfrak{S}_{\ell+1/2}$ be a complete Hecke eigenform. Then for any $\varepsilon > 0$ and all $x \geq 2$, we have*

$$(4.1) \quad \sum_{n \leq x} |\lambda_f(n)|^2 = D_f x + O_{f, \varepsilon}(x^{3/4+\varepsilon}),$$

where D_f is a positive constant depending on f .

Proof. We choose two smooth compactly supported functions w_\pm such that

- $w_-(x) = 1$ for $x \in [X + Y, 2X - Y]$, $w_-(x) = 0$ for $x \geq 2X$ and $x \leq X$;
- $w_+(x) = 1$ for $x \in [X, 2X]$, $w_+(x) = 0$ for $x \geq 2X + Y$ and $x \leq X - Y$;
- $w_\pm^{(j)}(x) \ll_j Y^{-j}$ for all $j \geq 0$;

- the Mellin transform of $w(x)$ is

$$\begin{aligned}
 \widehat{w}_{\pm}(s) &:= \int_0^{\infty} w_{\pm}(x) x^{s-1} dx \\
 (4.2) \quad &= \frac{1}{s \cdots (s+j-1)} \int_0^{\infty} w_{\pm}^{(j)}(x) x^{s+j-1} dx \\
 &\ll_j \frac{Y}{X^{1-\sigma}} \left(\frac{X}{|s|Y} \right)^j \quad \forall j \geq 1;
 \end{aligned}$$

- trivially $\widehat{w}_{\pm}(s) \ll X^{\sigma}$ and

$$(4.3) \quad \widehat{w}_{\pm}(1) = X + O(Y).$$

Obviously we have

$$(4.4) \quad \sum_n |\lambda_{\mathfrak{f}}(n)|^2 w_{-}(n) \leq \sum_{X < n \leq 2X} |\lambda_{\mathfrak{f}}(n)|^2 \leq \sum_n |\lambda_{\mathfrak{f}}(n)|^2 w_{+}(n).$$

Let the Dirichlet series associated with $|\lambda_{\mathfrak{f}}(n)|^2$ be defined as (see e.g. [13, (11)])

$$D(\mathfrak{f} \otimes \bar{\mathfrak{f}}, s) = \sum_{n=1}^{\infty} |\lambda_{\mathfrak{f}}(n)|^2 n^{-s}.$$

By the Mellin inversion formula

$$w_{\pm}(x) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \widehat{w}_{\pm}(s) x^{-s} ds,$$

we write

$$\sum_n |\lambda_{\mathfrak{f}}(n)|^2 w_{\pm}(n) = \frac{1}{2\pi i} \int_{(2)} \widehat{w}_{\pm}(s) D(\mathfrak{f} \otimes \bar{\mathfrak{f}}, s) ds.$$

With the help of Cauchy's residue theorem, we obtain that

$$(4.5) \quad \sum_n \lambda_{\mathfrak{f}}(n)^2 w_{\pm}(n) = D_{\mathfrak{f}} \widehat{w}_{\pm}(1) + \frac{1}{2\pi i} \int_{(\kappa)} \widehat{w}_{\pm}(s) D(\mathfrak{f} \otimes \bar{\mathfrak{f}}, s) ds,$$

where $\frac{1}{2} < \kappa < 1$ and $D_{\mathfrak{f}} := \text{Res}_{s=1} D(\mathfrak{f} \otimes \bar{\mathfrak{f}}, s)$. By (4.3), (4.2) with $j = 2$ and the convexity bound [13, Proposition 7]

$$D(\mathfrak{f} \otimes \bar{\mathfrak{f}}, s) \ll_{\mathfrak{f}, \varepsilon} (1 + |\tau|)^{2 \max(1-\sigma, 0) + \varepsilon} \quad \left(\frac{1}{2} < \sigma \leq 3 \right),$$

we derive

$$\sum_n |\lambda_{\mathfrak{f}}(n)|^2 w_{\pm}(n) = D_{\mathfrak{f}} X + O_{\mathfrak{f}, \varepsilon}(Y + X^{1+\kappa} Y^{-1}).$$

Taking $\kappa = \frac{1}{2} + \varepsilon$ and $Y = X^{3/4}$, and combining the obtained estimation with (4.4), we find that

$$\sum_{X < n \leq 2X} |\lambda_{\mathfrak{f}}(n)|^2 = D_{\mathfrak{f}} X + O_{\mathfrak{f}, \varepsilon}(X^{3/4+\varepsilon}),$$

which implies (4.1) after a dyadic summation. \square

Now we return to prove the theorem. Take $h = x^{\eta}$ where $\eta > \frac{3}{4}$ is specified later. Lemma 4.1 gives

$$(i) \quad Ch \leq \sum_{x \leq n \leq x+h} \lambda_{\mathfrak{f}}(n)^2 \quad \text{and} \quad (ii) \quad \sum_{x/m^2 \leq t \leq (x+h)/m^2} \lambda_{\mathfrak{f}}(n)^2 \ll hm^{-3/2}$$

for any $m \leq \sqrt{x+h}$, where the positive constant C and the implied \ll -constant depend on \mathfrak{f} and η only. Combining (i) with Lemma 3.1 leads to

$$Ch \leq \sum_{x \leq n \leq x+h} \lambda_{\mathfrak{f}}(n)^2 \leq C' \sum_{m \leq \sqrt{x+h}} \tau(m)^4 \sum_{x/m^2 \leq t \leq (x+h)/m^2}^b \lambda_{\mathfrak{f}}(t)^2$$

where \sum^b confines the running index over squarefree integers only and $C' > 0$ is a constant depending at most on \mathfrak{f} . By (ii) and the fact $\sum_{m \geq A} \tau(m)^4 m^{-3/2} \gg A^{-1/2+\varepsilon}$, we conclude that for a large enough constant A ,

$$\sum_{m \leq A} \tau(m)^4 \sum_{x/m^2 \leq t \leq (x+h)/m^2}^b \lambda_{\mathfrak{f}}(t)^2 \geq \{C/C' + O(A^{-1/2+\varepsilon})\}h \gg h$$

which is [13, (23)]. Thus, repeating the same argument (in [13, (24)-(26)]), we obtain [13, (26)] with a smaller admissible $h = x^\eta$ (here $\eta > \frac{3}{4}$ is required instead of $\eta > \frac{3}{4} + \varrho$).

Next we note that the new estimate (2.3) improves the upper bound $h^{3/4}x^\varepsilon$ in [13, (21) of Section 4.2] to $h^{1/2}x^\varepsilon$. Consequently, we get the new lower bound

$$x^{-1-\varrho}h^2 + O(h^{1/2}x^\varepsilon)$$

for [13, (27)]. The optimal choice of η is $\frac{2}{3}(1+\varrho) + \varepsilon$, and together with the constraint $\eta > \frac{3}{4}$, we choose

$$\eta = \max \left\{ \frac{2}{3}(1+\varrho), \frac{3}{4} \right\} + \varepsilon.$$

We complete the proof of Theorem 1 with the same argument in remaining part of [13, Section 4.2].

5. PREPARATION FOR THE TRUNCATED VORONOI FORMULA

Applying the additive character to replace the congruence condition, that is,

$$Q^{-1} \sum_{d|Q} \sum_{u \pmod{d}}^* e\left(\frac{u(n-a)}{d}\right) = \delta_{n \equiv a \pmod{Q}}$$

where $\delta_* = 1$ if $*$ holds and 0 otherwise, we have

$$(5.1) \quad \mathcal{S}_{\mathfrak{f}}^A(x) := \sum_{\substack{n \leq x \\ n \equiv a \pmod{Q}}} \lambda_{\mathfrak{f}}(n) = Q^{-1} \sum_{d|Q} \mathcal{S}_{\mathfrak{f}}(x, a/d),$$

where

$$(5.2) \quad \mathcal{S}_{\mathfrak{f}}(x, a/d) := \sum_{u \pmod{d}}^* e\left(\frac{-au}{d}\right) \sum_{n \leq x} \lambda_{\mathfrak{f}}(n) e\left(\frac{nu}{d}\right).$$

Here $\sum_{u \pmod{d}}^*$ denotes the sum over $u \pmod{d}$ with $(u, d) = 1$. The inner sum over n is clearly associated with $L_{\mathfrak{f}}(s, u/d)$, thus we introduce the auxiliary function

$$(5.3) \quad \mathcal{L}_{\mathfrak{f}}(s, a/d) := \sum_{u \pmod{d}}^* e\left(-\frac{au}{d}\right) L_{\mathfrak{f}}(s, u/d).$$

The Dirichlet series associated to $\mathcal{S}^A(x)$,

$$(5.4) \quad L_{\mathfrak{f}}(s, a, Q) := \sum_{\substack{n \geq 1 \\ n \equiv a \pmod{Q}}} \lambda_{\mathfrak{f}}(n) n^{-s}$$

is equal to

$$(5.5) \quad L_f(s, a, Q) = Q^{-1} \sum_{d|Q} \mathcal{L}_f(s, a/d).$$

Plainly $\mathcal{L}_f(s, a/d)$ satisfies a functional equation by (3.7),

$$(5.6) \quad q_d^s L_\infty(s) \mathcal{L}_f(s, a/d) = i^{-(\ell+1/2)} q_d^{1-s} L_\infty(1-s) \tilde{\mathcal{L}}_f(1-s, a/d)$$

where $\tilde{L}_f(s, v/d)$ is defined as in (3.9) and

$$\tilde{\mathcal{L}}_f(s, a/d) = \sum_{u \pmod{d}}^* e\left(-\frac{au}{d}\right) \tilde{L}_f(s, \bar{u}/d) \quad (u\bar{u} \equiv 1 \pmod{d}).$$

When $\Re s > 1$, we may express $\tilde{\mathcal{L}}_f(s, a/d)$ as a Dirichlet series whose coefficients are products of $\lambda(n; d)$ and the Kloosterman-Salié sums. Indeed, by (3.9), we have

$$(5.7) \quad \tilde{\mathcal{L}}_f(s, a/d) = \sum_{n \geq 1} \lambda(n; d) K(a, n; d) n^{-s}$$

where (noting $v = \bar{u} \pmod{d}$),

$$(5.8) \quad K(a, n; d) := \sum_{u \pmod{d}}^* \varpi_d(n, \bar{u}) e\left(-\frac{au}{d}\right).$$

By (3.10),

$$K(a, n; d) = \begin{cases} \sum_{u \pmod{d}}^* \varepsilon_u^{2\ell+1} \left(\frac{d}{u}\right) e\left(-\frac{a\bar{u} + nu}{4d}\right) & \text{if } 4 \mid d, \\ \sum_{u \pmod{d}}^* \varepsilon_u^{2\ell+1} \left(\frac{d}{u}\right) e\left(-\frac{4a\bar{u} + nu}{4d}\right) & \text{if } 2 \parallel d, \\ i^{\ell+1/2} \varepsilon_d^{-(2\ell+1)} \sum_{u \pmod{d}}^* \left(\frac{u}{d}\right) e\left(-\frac{a\bar{u} + 4nu}{d}\right) & \text{if } 2 \nmid d. \end{cases}$$

Lemma 5.1. *Let $\tau(d)$ be the divisor function. We have*

$$(5.9) \quad |K(a, n; d)| \ll (d, n)^{1/2} d^{1/2} \tau(d).$$

Moreover, for the case $2 \nmid d$, if there exists $x \in \{a, n\}$ such that $(x, d) = 1$, then

$$(5.10) \quad K(a, n; d) = i^{\ell+1/2} \varepsilon_d^{-2\ell} d^{1/2} \left(\frac{x}{d}\right) \sum_{y^2 \equiv an \pmod{d}} e\left(\frac{y}{d}\right).$$

Proof. We express $K(a, n; d)$ in terms of Kloosterman-Salié sums (see Appendix for their definitions), as follows:

$$(5.11) \quad K(a, n; d) = \begin{cases} \overline{K_{2\ell+1}(n, a; d)} & \text{for } 4 \mid d, \\ \frac{1}{4} \overline{K_{2\ell+1}(n, a; 4d)} & \text{for } 2 \parallel d, \\ i^{\ell+1/2} \varepsilon_d^{-(2\ell+1)} \overline{S(4n, a; d)} & \text{for } 2 \nmid d, \end{cases}$$

where in the case of $2 \parallel d$, the range of summation is enlarged to a reduced residue system $\pmod{4d}$. From (9.2) below, we have

$$(5.12) \quad |K(a, n; d)| \ll (d, n)^{1/2} d^{1/2} \tau(d).$$

The formula (5.10) follows from the result in [9, Lemma 4.9] for the Salié sum. \square

Lemma 5.2. *Let $d \geq 1$ and a be any integers. For any $\varepsilon > 0$, we have*

$$(5.13) \quad \mathcal{L}_{\mathfrak{f}}(\sigma + i\tau, a/d) \ll d^{(3-\sigma)/2+2\varepsilon} (1 + |\tau|)^{1-\sigma+2\varepsilon} \quad (-\varepsilon \leq \sigma \leq 1 + \varepsilon, \tau \in \mathbb{R}),$$

where the implied \ll -constant depends on \mathfrak{f} and ε only.

Proof. Let $\Re s = 1 + \varepsilon$. By (3.3) and (3.6), we have trivially $L_{\mathfrak{f}}(s, u/d) \ll_{\varepsilon} 1$ and with (5.3), $\mathcal{L}_{\mathfrak{f}}(s, a/d) \ll_{\varepsilon} d$. Next for $\Re s = -\varepsilon$, we infer from (5.6) and (5.7) that

$$\mathcal{L}_{\mathfrak{f}}(s, a/d) = i^{-(\ell+1/2)} q_d^{1-2s} \frac{L_{\infty}(1-s)}{L_{\infty}(s)} \sum_{n \geq 1} \frac{\lambda(n; d) K(a, n; d)}{n^{1-s}}.$$

Thus, with (5.12) and Stirling's formula, it follows that

$$\begin{aligned} \mathcal{L}_{\mathfrak{f}}(-\varepsilon + i\tau, a/d) &\ll (d^{3/2}(1 + |\tau|))^{1+\varepsilon} \sum_{n \geq 1} |\lambda(n; d)| (n, d)^{1/2} n^{-(1+\varepsilon)} \\ &\ll (d^{3/2}(1 + |\tau|))^{1+\varepsilon} \end{aligned}$$

because $|\lambda(n; d)| (n, d)^{1/2} \leq |\lambda(n; d)|^2 + (n, d)$, implying that the last summation is

$$\ll \sum_{n \geq 1} |\lambda(n; d)|^2 n^{-(1+\varepsilon)} + \sum_{l|d} l^{-\varepsilon} \sum_{n \geq 1} n^{-(1+\varepsilon)} \ll \tau(d).$$

An application of Phragmén–Lindelöf principle completes the proof. \square

6. TRUNCATED VORONOI FORMULA

This section is devoted to the Voronoi formulas. In order for a simpler form for the result, let us set, with the notation (5.8),

$$(6.1) \quad \phi_a(n, d) := \sqrt{q_d} i^{-(\ell+1/2)} K(a, n; d) \ll (n, d)^{1/2} \tau(d) d$$

by (5.12), and trivially $|\phi_a(n, d)| \leq \sqrt{2} d^{3/2}$. We have the following result.

Theorem 3. *Let $\ell \geq 2$ be an integer and $\mathfrak{f} \in \mathfrak{S}_{\ell+1/2}$ be an eigenform of all Hecke operators. Then for any $\varepsilon > 0$, we have*

$$(6.2) \quad \begin{aligned} \mathcal{S}_{\mathfrak{f}}(x, a/d) &= \frac{x^{1/4}}{\pi \sqrt{2}} \sum_{n \leq M} \frac{\lambda(n; d) \phi_a(n, d)}{n^{3/4}} \cos \left(4\pi \frac{\sqrt{nx}}{q_d} - \frac{\ell+1}{2} \pi \right) \\ &\quad + O_{\mathfrak{f}, \varepsilon} (x^{\varepsilon} d^2 (x^{1/2+\varrho} M^{-1/2} + M^{\varrho})) \end{aligned}$$

uniformly for $2 \leq M \leq x$ and $1 \leq d \leq x^{1/2}$, where ϱ is defined as in (3.5).

Moreover for $1 \leq Q \leq x^{1/2}$ and any integer a ,

$$\begin{aligned} \mathcal{S}_{\mathfrak{f}}^A(x) &= \frac{x^{1/4}}{\sqrt{2} \pi Q} \sum_{d|Q} \sum_{n \leq M} \frac{\lambda(n; d) \phi_a(n, d)}{n^{3/4}} \cos \left(4\pi \frac{\sqrt{nx}}{q_d} - \frac{\ell+1}{2} \pi \right) \\ &\quad + O(x^{\varepsilon} Q (x^{1/2+\varrho} M^{-1/2} + M^{\varrho})). \end{aligned}$$

In particular, for $Q \leq x^{\frac{1}{2}-\varrho}$ and any a ,

$$(6.3) \quad \mathcal{S}_{\mathfrak{f}}^A(x) \ll_{\mathfrak{f}, \varepsilon} Q^{1/3} x^{(1+\varrho)/3+\varepsilon}.$$

Remark 3. It is shown in [15, Proposition 3.2] that $\mathcal{S}_{\mathfrak{f}}^{\mathbb{N}}(x) \ll x^{2/5+\varepsilon}$, which is superseded by the particular case $\mathcal{A} = \mathbb{N}$ (and $Q = 1$) of (6.3) for $\varrho = 1/6 + \varepsilon$ is admissible.

Proof. Let $d \leq x^{1/2}$, $1 \leq M \leq x$ and $T > 1$ be chosen as

$$(6.4) \quad T^2 = q_d^{-2} 4\pi^2 (M + 1/2)x \gg 1.$$

We apply the Perron formula (cf. [16, Corollary II.2.2.1]) to (5.3) with $\kappa := 1 + \varepsilon$, $\sigma_a = \alpha = 1$ and $B(n) = C_\varepsilon n^\varepsilon$ to write

$$(6.5) \quad \mathcal{S}_{\mathfrak{f}}(x, a/d) = \frac{1}{2\pi i} \int_{\kappa-iT}^{\kappa+iT} \mathcal{L}_{\mathfrak{f}}(s, a/d) \frac{x^s}{s} ds + O_{\mathfrak{f}, \varepsilon} \left(\frac{dx^{1+\varepsilon}}{T} \right).$$

We deform the line of integration to the contour \mathcal{L} joining the points $\kappa - iT$, $-\varepsilon - iT$, $-\varepsilon + iT$, $\kappa + iT$. Let $\mathcal{L}_v := [-\varepsilon - iT, -\varepsilon + iT]$. By Lemma 5.2, the integrals over the horizontal segments of \mathcal{L} are $\ll x^\varepsilon (xT^{-1} + d^{3/2})$, and the pole of the integrand at $s = 0$ gives $\mathcal{L}_{\mathfrak{f}}(0, a/d) \ll d^{3/2+\varepsilon}$. By the functional equation (5.6), the integral over \mathcal{L}_v equals

$$\frac{1}{2\pi i} \int_{\mathcal{L}_v} \mathcal{L}_{\mathfrak{f}}(s, a/d) \frac{x^s}{s} ds = q_d i^{-(\ell+1/2)} \frac{1}{2\pi i} \int_{\mathcal{L}_v} \frac{L_\infty(1-s)}{L_\infty(s)} \tilde{\mathcal{L}}_{\mathfrak{f}}(1-s, a/d) \left(\frac{\sqrt{x}}{q_d} \right)^{2s} \frac{ds}{s}$$

By (5.7) and (6.1), we express (6.5) into

$$(6.6) \quad \mathcal{S}_{\mathfrak{f}}(x, a/d) = \frac{\sqrt{q_d}}{2\pi} \sum_{n \geq 1} \frac{\lambda(n; d) \phi_a(n, d)}{n} I_{\mathcal{L}_v} \left(\frac{2\pi \sqrt{nx}}{q_d} \right) + O \left(\frac{dx^{1+\varepsilon}}{T} + d^{3/2} x^\varepsilon \right)$$

where

$$I_{\mathcal{L}_v}(y) := \frac{1}{2\pi i} \int_{\mathcal{L}_v} \frac{\Gamma(1-s+\ell/2-1/4)}{\Gamma(s+\ell/2-1/4)} \cdot \frac{y^{2s}}{s} ds.$$

Next we apply the stationary phase method to bound $I_{\mathcal{L}_v}(y)$ for large y and give an asymptotic expansion in terms of trigonometric functions for small y .

With Stirling's formula, for $\tau > 0$, the integrand equals

$$e^{i\pi(\ell-1)/2} y^{2\sigma} \tau^{-2\sigma} e^{2i\tau \log(ey/\tau)} \{1 + c_1 \tau^{-1} + O(\tau^{-2})\}$$

for any $|\tau| \geq 1$ and $|\sigma| \leq A$, where c_1 and $A > 0$ denote some suitable constants and the implied O -constant is independent of τ and y . Set $g(\tau) := 2\tau \log(ey/\tau)$, then $g'(\tau) = 2 \log(y/\tau)$. With the second mean value theorem for integrals (cf. [16, Theorem I.0.3]), we obtain for $y > T$ and $\sigma = -\varepsilon$,

$$(6.7) \quad \int_1^T y^{2\sigma} \tau^{-2\sigma} e^{ig(\tau)} \{1 + c_1 \tau^{-1} + O(\tau^{-2})\} d\tau \ll T^{2\varepsilon} y^{2\sigma} \left| \log \frac{y}{T} \right|^{-1} + T^{2\varepsilon-1} y^{2\sigma},$$

and for $y < T$ and $\sigma = \frac{1}{2} + \varepsilon$,

$$(6.8) \quad \int_T^\infty y^{2\sigma} \tau^{-2\sigma} e^{ig(\tau)} \{1 + c_1 \tau^{-1} + O(\tau^{-2})\} d\tau \ll T^{-1-2\varepsilon} y^{2\sigma} \left| \log \frac{y}{T} \right|^{-1} + T^{-1-2\varepsilon} y^{2\sigma}.$$

For $n > M$, we infer by (6.7) that

$$I_{\mathcal{L}_v} \left(\frac{2\pi \sqrt{nx}}{q_d} \right) \ll_k \left(\frac{x}{\sqrt{n}} \right)^{2\varepsilon} \left(\left| \log \frac{n}{M+1/2} \right|^{-1} + d(Mx)^{-1/2} \right).$$

By $\lambda(n; d) \ll n^{\varrho+\varepsilon}$ from Lemma 3.1 and $|\phi_a(n, d)| \leq \sqrt{2}d^{3/2}$, it follows that

$$\begin{aligned} \sqrt{q_d} \sum_{n>M} \frac{|\lambda(n; d)\phi_a(n, d)|}{n^{1+\varepsilon}} \left| \log \frac{n}{M+1/2} \right|^{-1} &\ll d^2 M^\varrho \sum_{M < n < 2M} |n - (M+1/2)|^{-1} \\ &\ll d^2 M^{\varrho+\varepsilon}. \end{aligned}$$

Consequently we deduce that

$$(6.9) \quad \frac{\sqrt{q_d}}{2\pi} \sum_{n>M} \frac{\lambda_{\mathfrak{h}}(n)\phi_a(n, d)}{n} I_{\mathcal{L}_v} \left(\frac{2\pi\sqrt{nx}}{q_d} \right) \ll x^\varepsilon d^2 M^\varrho + x^\varepsilon d^2 (Mx)^{-1/2}.$$

For $n \leq M$, we complete the path \mathcal{L}_v to the contour \mathcal{L}_v^* so as to apply [1, Lemma 1], where \mathcal{L}_v^* is the positively oriented contour consisting of \mathcal{L}_v , \mathcal{L}_v^\pm and \mathcal{L}_h^\pm with

$$\mathcal{L}_v^\pm := [\tfrac{1}{2} + \varepsilon \pm iT, \tfrac{1}{2} + \varepsilon \pm i\infty), \quad \mathcal{L}_h^\pm := [-\varepsilon \pm iT, \tfrac{1}{2} + \varepsilon \pm iT].$$

Correspondingly we denote by $I_{\mathcal{L}_v^\pm}$ and $I_{\mathcal{L}_h^\pm}$ the integrals over these segments. By (6.8), the integral over the vertical line segments \mathcal{L}_v^\pm is

$$I_{\mathcal{L}_v^\pm} \ll x^\varepsilon \left(\frac{n}{M} \right)^{1/2} \left| \log \frac{n}{M+1/2} \right|^{-1},$$

while for the horizontal segments, $I_{\mathcal{L}_h^\pm}$ contributes at most $O((n/M)^\varepsilon)$. Thus

$$\begin{aligned} (6.10) \quad &\frac{\sqrt{q_d}}{2\pi} \sum_{n \leq M} \frac{\lambda(n; d)\phi_a(n, d)}{n} (I_{\mathcal{L}_v^\pm} + I_{\mathcal{L}_h^\pm}) \\ &\ll x^\varepsilon d^2 M^{\varrho-1/2} \sum_{M/2 \leq n \leq M} n^{-1/2} \left| \log \frac{M+1/2}{M+1/2-n} \right|^{-1} \\ &\ll x^\varepsilon d^2 M^\varrho. \end{aligned}$$

Inserting (6.10) and (6.9) into (6.6), we get from our choice of T ,

$$\begin{aligned} (6.11) \quad \mathfrak{S}_{\mathfrak{f}}(x, a/d) &= \frac{\sqrt{q_d}}{2\pi} \sum_{1 \leq n \leq M} \frac{\lambda(n; d)\phi_a(n, d)}{n} I_{\mathcal{L}_v^*} \left(\frac{2\pi\sqrt{nx}}{q_d} \right) \\ &\quad + O(x^\varepsilon d^2 (x^{1/2+\varrho} M^{-1/2} + M^\varrho)). \end{aligned}$$

Now all the poles of the integrand in

$$I_{\mathcal{L}_v^*}(y) := \frac{1}{2\pi i} \int_{\mathcal{L}_v^*} \frac{\Gamma(1-s+\ell/2-1/4)\Gamma(s)}{\Gamma(s+\ell/2-1/4)\Gamma(s+1)} y^{2s} ds.$$

lie on the right of the contour \mathcal{L}_v^* . After a change of variable s into $1-s$, we have

$$I_{\mathcal{L}_v^*}(y) = \frac{1}{\pi} I_0(y^2),$$

with

$$I_0(y) := \frac{1}{2\pi i} \int_{\mathcal{L}_\varepsilon} \frac{\Gamma(s+(2\ell-1)/4)\Gamma(1-s)}{\Gamma(1-s+(2\ell-1)/4)\Gamma(2-s)} y^{1-s} ds.$$

Here \mathcal{L}_ε consists of the line $s = \frac{1}{2} - \varepsilon + i\tau$ with $|\tau| \geq T$, together with three sides of the rectangle whose vertices are $\frac{1}{2} - \varepsilon - iT$, $1 + \varepsilon - iT$, $1 + \varepsilon + iT$ and $\frac{1}{2} - \varepsilon + iT$. Clearly our I_0 is a particular case of I_ρ defined in [1, Lemma 1], corresponding to the choice of

parameters $A = \delta = N = \omega = \alpha_1 = 1$, $\beta_1 = \mu = (\ell - 2)/4$, $\rho = m = 0$, $a = -\frac{3}{4}$, $c_0 = \frac{1}{2}$, $h = 2$, $k_0 = -(\ell + 1)/2$. It hence follows that

$$(6.12) \quad I_{\mathcal{L}_v^*} \left(\frac{2\pi\sqrt{nx}}{q_d} \right) = e'_0 \sqrt{\frac{2\pi}{q_d}} (nx)^{1/4} \cos \left(4\pi \frac{\sqrt{nx}}{q_d} - \frac{\ell+1}{2} \pi \right) + O(d^{1/2}(nx)^{-1/4}).$$

The value of e'_0 [1, Lemma 1] is $1/\sqrt{\pi}$, and the main term in (6.2) follows from (6.12) and (6.11). With a simple checking, the O -term in (6.12) gives a term that will be absorbed in (6.11).

Finally we set $M = Q^{4/3}x^{(1+4\rho)/3}$ and note from (6.1) that

$$\sum_{n \leq M} \frac{|\lambda(n; d)\phi_a(n, d)|}{n^{3/4}} \ll d^{1+\varepsilon} \sum_{n \leq M} |\lambda(n; d)|^2 n^{-3/4} + d^{1+\varepsilon} \sum_{n \leq M} (n, d) n^{-3/4},$$

which is $\ll x^\varepsilon d M^{1/4}$ with (3.3). \square

7. PREPARATION FOR THE PROOF OF THEOREM 2

We consider odd Q only, then $q_d = 2d$ and $\lambda(n; d) = \lambda_{\mathfrak{h}}(n)$ for all $d \mid Q$. The idea of proof is the same as in Heath-Brown & Tsang [5], however, some new technicality arises because of the new frequencies (\sqrt{n}/q_d rather than \sqrt{n}). Consequently, instead of $\sqrt{1}$, we shall apply their argument to the frequency $\sqrt{n_0}/Q$ where $n_0 = 2^j f_0$ with $j \geq 0$ and f_0 squarefree, and simultaneously, require the coefficient $\lambda_{\mathfrak{h}}(n_0)\phi_a(n_0, Q)$ to be non-vanishing. We can guarantee the existence of n_0 under certain circumstances.

For convenience, let us recall our notation (specialized to this case $2 \nmid d$):

$$\mathcal{S}_{\mathfrak{f}}^A(x) = \sum_{\substack{n \leq x \\ n \equiv a \pmod{Q}}} \lambda_{\mathfrak{f}}(n) \quad \text{and} \quad \mathcal{S}_{\mathfrak{f}}(x, a/d) := \sum_{n \leq x} \lambda_{\mathfrak{f}}(n) R_d(n - a).$$

where $R_d(m) = \sum_{u \pmod{d}}^* e(mu/d)$ is the Ramanujan sum. Their associated Dirichlet series are

$$L_{\mathfrak{f}}(s, a, Q) := \sum_{\substack{n \geq 1 \\ n \equiv a \pmod{Q}}} \lambda_{\mathfrak{f}}(n) n^{-s} \quad \text{and} \quad \mathcal{L}_{\mathfrak{f}}(s, a/d) := \sum_{n \geq 1} \lambda_{\mathfrak{f}}(n) R_d(n - a) n^{-s}.$$

Moreover, $L_{\mathfrak{f}}(s, a, Q) = Q^{-1} \sum_{d \mid Q} \mathcal{L}_{\mathfrak{f}}(s, a/d)$ and

$$(2d)^s L_{\infty}(s) \mathcal{L}_{\mathfrak{f}}(s, a/d) = i^{-(\ell+1/2)} (2d)^{1-s} L_{\infty}(1-s) \tilde{\mathcal{L}}_{\mathfrak{f}}(1-s, a/d)$$

where

$$\tilde{\mathcal{L}}_{\mathfrak{f}}(s, a/d) := \sum_{n \geq 1} \lambda_{\mathfrak{h}}(n) K(a, n; d) n^{-s}.$$

Lemma 7.1. *Under the assumption that $\{\lambda_{\mathfrak{f}}(n)\}_{n \in \mathbb{N}}$ is a real sequence, for all a, d , the sequences $\{i^{-(\ell+1/2)} \lambda_{\mathfrak{h}}(n) K(a, n; d)\}_{n \in \mathbb{N}}$ are real.*

Proof. Since the Ramanujan sum $R_d(m)$ is real-valued, $\mathcal{L}_{\mathfrak{f}}(s, a/d)$ is real-valued for $s \in (1, \infty)$ under the given assumption. The holomorphicity of $\mathcal{L}_{\mathfrak{f}}(s, a/d)$ implies that $\overline{\mathcal{L}_{\mathfrak{f}}(\bar{s}, a/d)}$ is holomorphic. Thus $\overline{\mathcal{L}_{\mathfrak{f}}(\bar{s}, a/d)} = \mathcal{L}_{\mathfrak{f}}(s, a/d)$ on \mathbb{C} (as they are equal on $(1, \infty)$). The lemma follows. \square

Lemma 7.2. *When the sequence $\{\lambda_f(n)\}_{n \in A}$ contains nonzero terms, the function $\mathcal{L}_f(s, a/d)$ is non-identically zero for all $d \mid Q$.*

Proof. Suppose not, say, $\mathcal{L}_f(s, a/d_0) \equiv 0$. Then

$$\sum_{\substack{n \geq 1 \\ n \equiv a \pmod{Q}}} \lambda_f(n) n^{-s} = Q^{-1} \sum_{\substack{d \mid Q \\ d \neq d_0}} \mathcal{L}_f(s, a/d) = \sum_{n \geq 1} n^{-s} \lambda_f(n) Q^{-1} \sum_{\substack{d \mid Q \\ d \neq d_0}} R_d(n - a).$$

With the standard formula for the Ramanujan sum, we infer that

$$\delta_{n \equiv a \pmod{Q}} \lambda_f(n) = \lambda_f(n) Q^{-1} \sum_{\substack{d \mid Q \\ d \neq d_0}} \sum_{\substack{\delta \mid d \\ (d/\delta) \mid (n-a)}} \mu(\delta) (d/\delta) \quad \forall n \geq 1.$$

Take $n \equiv a \pmod{Q}$ such that $\lambda_f(n) \neq 0$. We obtain that

$$Q - \phi(d_0) = \sum_{\substack{d \mid Q \\ d \neq d_0}} \phi(d) = \sum_{\substack{d \mid Q \\ d \neq d_0}} \sum_{\delta \mid d} \mu(\delta) (d/\delta) = Q.$$

Contradiction arises. \square

Proposition 1. *Let $Q \geq 1$ be odd and $0 \leq a < d$. Suppose $n_0 = 2^j f_0$ with f_0 squarefree and $j \geq 0$ is an integer such that*

$$(7.1) \quad \lambda_h(n_0) \phi_a(n_0, Q) \neq 0.$$

Then there are constants $c_0 = c_0(f, Q, n_0)$ and $x_0 = x_0(f, Q, n_0)$ such that $S_f^A(x)$ attains at least one sign change in the interval $[x, x + c_0 \sqrt{x}]$ for all $x \geq x_0$.

Proof. Let α a parameter determined later and T be any sufficiently large number. Set

$$F_f(t + \alpha u) := \pi \sqrt{Q} \frac{S_f^A((Q(t + \alpha u))^2)}{\sqrt{t + \alpha u}} \quad (t \in [T, 2T], u \in [-1, 1]).$$

By Theorem 3 with $M = (QT)^2$, we deduce that

$$\begin{aligned} F_f(t + \alpha u) &= \sum_{d \mid Q} \sum_{n \leq (QT)^2} \frac{\lambda_h(n) \phi_a(n, d)}{n^{3/4}} \cos \left(\pi(t + \alpha u) \frac{Q\sqrt{n}}{d} - \frac{\ell + 1}{2} \pi \right) \\ &\quad + O(Q(QT)^{2\varrho - 1/2 + \varepsilon}). \end{aligned}$$

Let $\tau = 1$ or -1 , and define

$$k_\tau(u) := (1 - |u|)(1 + \tau \cos(2\pi \alpha \sqrt{n_0} u)).$$

Then as in the proof of [12, Lemma 3.2], for any $n \in \mathbb{N}$ and $t \in \mathbb{R}$, the integral

$$r_n = r_n(\alpha, \tau, t) := \int_{-1}^1 k_\tau(u) \cos \left(2\pi(t + \alpha u) \frac{Q\sqrt{n}}{d} - \frac{\ell + 1}{2} \pi \right) du$$

satisfies

$$(7.2) \quad \begin{aligned} r_n &= \delta_{Q\sqrt{n}=d\sqrt{n_0}} \cdot \frac{\tau}{2} \cos \left(2\pi t \sqrt{n_0} - \frac{\ell + 1}{2} \pi \right) \\ &\quad + O \left(\min \left(1, \frac{1}{\alpha^2 n} \right) + \delta_{Q\sqrt{n} \neq d\sqrt{n_0}} \min \left(1, \frac{1}{(\alpha_{n,d}^-)^2} \right) \right), \end{aligned}$$

where $\alpha_{n,d}^- = \alpha |Q\sqrt{n} - d\sqrt{n_0}|/d$, $\delta_* = 1$ if $*$ holds, or 0 otherwise. The O -constant is absolute.

Observe that $Q\sqrt{n} = d\sqrt{n_0}$ if and only if $2^j f_0 = (Q/d)^2 n$ which is equivalent to $n = 2^j f_0 = n_0$ and $d = Q$ since f_0 is squarefree and Q/d is odd. Following from (7.2) and (7.2), the integral

$$J_\tau(t) = \int_{-1}^1 F_{\mathfrak{f}}(t + \alpha u) k_\tau(u) du$$

can be written as

$$(7.3) \quad J_\tau(t) = \frac{\tau}{2} \frac{\lambda_{\mathfrak{h}}(n_0) \phi_a(n_0, Q)}{n_0^{3/4}} \cos \left(2\pi t \sqrt{n_0} - \frac{\ell+1}{2} \pi \right) + E + O(Q(QT)^{2\varrho-1/2+\varepsilon})$$

where

$$E \ll \frac{1}{\alpha^2} \sum_{d|Q} \sum_{n \leq (QT)^2} \frac{|\lambda_{\mathfrak{h}}(n) \phi_a(n, d)|}{n^{7/4}} + \sum_{d|Q} \frac{d^2}{\alpha^2} \sum_{\substack{n \leq (QT)^2 \\ Q\sqrt{n} \neq d\sqrt{n_0}}} \frac{|\lambda_{\mathfrak{h}}(n) \phi_a(n, d)|}{n^{3/4} |Q\sqrt{n} - d\sqrt{n_0}|^2}.$$

Using the bounds $\phi_a(n, d) \ll d^{3/2}$ and $\lambda_{\mathfrak{h}}(n) \ll n^{\varrho}$, a little calculation gives

$$E \ll Q^3 n_0^{\varrho+1/4} \alpha^{-2}.$$

Let $A_0 := |\lambda_{\mathfrak{h}}(n_0) \phi_a(n_0, Q)| n_0^{-3/4}$, which is > 0 . Fix a sufficiently large $\alpha = \alpha(\mathfrak{f}, n_0, Q)$, so that E is $< \frac{1}{8} A_0$, and then a sufficiently large $T_0 = T_0(\mathfrak{f}, n_0, Q, \alpha)$ such that the O -term $O(Q(QT)^{2\varrho-1/2+\varepsilon})$ is $\leq \frac{1}{8} A_0$ for all $T \geq T_0$. Now observe that for any $m \in \mathbb{N}$, the absolute value of the cosine factor is $1/\sqrt{2}$ if $t = t_m$ where

$$t_m := (m + \frac{1}{8}) n_0^{-1/2}.$$

This implies $|J_\tau(t_m)| > \frac{1}{4}(\sqrt{2} - 1) A_0 > 0$ whenever $t_m > T_0 + \alpha$. Since $J_\pm(t_m)$ are of opposite signs and the kernel function k_τ is nonnegative, there is a pair of $t_m^\pm \in [t_m - \alpha, t_m + \alpha]$ for which $\pm F_{\mathfrak{f}}(t_m^\pm) > 0$. Equivalently, $\mathcal{S}_{\mathfrak{f}}^A(y)$ attains a sign change in every interval of the form $[(Q(t_m - \alpha))^2, (Q(t_m + \alpha))^2]$ whose length is $\ll \alpha(Q^2 t_m) \ll_{\mathfrak{f}, Q, n_0} \sqrt{x}$ when $x = (Q t_m)^2$. Our result follows readily. \square

8. PROOF OF THEOREM 2

In view of Proposition 1, the main task is to study the condition $\lambda_{\mathfrak{h}}(n_0) \phi_a(n_0, Q)$. Recall $\phi_a(n, Q) = \sqrt{2Q} i^{-(\ell+1/2)} K(a, n; Q)$ by (6.1). Clearly, $\phi_a(n, 1) = \sqrt{2}$. In general, we have by Lemma 9.1 (2),

$$(8.1) \quad \phi_a(n, Q) = \sqrt{2Q} \varepsilon_Q^{-(2\ell+1)} \prod_{p^\alpha \parallel Q} S(n \overline{4Q_p}, a \overline{Q_p}; p^\alpha)$$

where $S(m, n; c)$ is defined as in (9.1), $Q_p = Q/p^\alpha$ and $\bar{x}x \equiv 1 \pmod{p^\alpha}$ for each term inside the product, $\forall p^\alpha \parallel Q$.

♠ Case 1. $Q = 1$. It suffices to find a squarefree t and a $j \geq 0$ such that $\lambda_{\mathfrak{h}}(2^j t) \neq 0$. By Lemma 7.2, $\mathcal{L}_{\mathfrak{f}}(s, 1)$ and thus $\tilde{\mathcal{L}}_{\mathfrak{f}}(s, 1) = \sum_{n \geq 1} \lambda_{\mathfrak{h}}(n) n^{-s}$ are not identical to the zero function. Thus $\lambda_{\mathfrak{h}}(n) \neq 0$ for some $n \in \mathbb{N}$. Write $n = 2^j t m^2$ where t is squarefree and m is odd, $\lambda_{\mathfrak{h}}(2^j t) \neq 0$ from (3.4).

- ♠ Case 2. $a = 0$ and $p^\alpha \parallel Q$ implies α being odd. By Lemma 9.1 (2)-(3) and (8.1), $\phi_0(n, Q) = 0$ if $(n, Q) > 1$. Repeating the argument in Case 1, we get $\lambda_{\mathfrak{h}}(n)\phi_0(n, Q) \neq 0$ for some $n \in \mathbb{N}$. This n has to be coprime with Q . Write $n = 2^j t m^2$ with squarefree t and odd m , then $\lambda_{\mathfrak{h}}(2^j t) \neq 0$ (from $\lambda_{\mathfrak{h}}(2^j t m^2) \neq 0$) and $\phi_0(2^j t, Q) \neq 0$ because

$$S(hk, 0; Q) = \left(\frac{h}{Q}\right) S(k, 0; Q)$$

if $(h, Q) = 1$, from the definition of the Salié sum.

- ♠ Case 3. $(a, Q) = 1$ and $p^2 \mid Q, \forall p \mid Q$. The argument is similar to the previous cases – firstly finding $n = 2^j t m^2$, with squarefree t and odd m , for which $\lambda_{\mathfrak{h}}(n)\phi_0(n, Q) \neq 0$. But now we need (5.10) to analyze the Salié sum, which gives

$$\phi_a(2^j t m^2, Q) = \sqrt{2} Q \varepsilon_Q^{-2\ell} \left(\frac{a}{Q}\right) c_{a2^j t}(m, Q)$$

where

$$(8.2) \quad c_b(m, d) = \sum_{\substack{y \pmod{d} \\ y^2 \equiv b m^2 \pmod{d}}} e\left(\frac{y}{d}\right).$$

As in (8.1), we have the factorization

$$c_{a2^j t}(m, Q) = \prod_{p^\alpha \parallel Q} c_{\overline{Q_p} a 2^j t}(m, p^\alpha)$$

and the lemma below assures $(m, Q) = 1$ and $\phi_a(2^j t, Q) \neq 0$ when $\phi_a(2^j t m^2, Q) \neq 0$. Hence this case is also complete.

Lemma 8.1. *Let $b \in \mathbb{Z}$, p an odd prime and $\alpha \geq 2$. Define $c_b(m, p^\alpha)$ as in (8.2). Then*

- (i) $c_b(m, p^\alpha) = 0$ if $p \mid m$, and
- (ii) $c_b(1, p^\alpha) \neq 0$ if $c_b(m, p^\alpha) \neq 0$ with $p \nmid m$.

Proof. (i) Write $m = p^\beta m'$ where $p \nmid m'$.

- $\alpha = 2\gamma \leq 2\beta$. Then

$$c_b(m, p^\alpha) = \sum_{y^2 \equiv 0 \pmod{p^\alpha}} e\left(\frac{y}{p^\alpha}\right) = \sum_{l \pmod{p^\gamma}} e\left(\frac{l}{p^\gamma}\right) = 0.$$

- $\alpha = 2\gamma + 1 \leq 2\beta$. Then y is of the form $y = l p^{\gamma+1}$, and as $\gamma \geq 1$,

$$c_b(m, p^\alpha) = \sum_{y^2 \equiv 0 \pmod{p^\alpha}} e\left(\frac{y}{p^\alpha}\right) = \sum_{l \pmod{p^\gamma}} e\left(\frac{l}{p^\gamma}\right) = 0.$$

- $\alpha > 2\beta \geq 2$. Then $y = lp^\beta$ and thus

$$\begin{aligned}
c_b(m, p^\alpha) &= \sum_{l^2 \equiv bm'^2 \pmod{p^{\alpha-2\beta}}} \sum_{y \equiv p^\beta l \pmod{p^\alpha}} e\left(\frac{y}{p^\alpha}\right) \\
&= \sum_{l^2 \equiv bm'^2 \pmod{p^{\alpha-2\beta}}} \sum_{t \pmod{p^\beta}} e\left(\frac{l + tp^{\alpha-2\beta}}{p^{\alpha-\beta}}\right) \\
&= \sum_{l^2 \equiv bm'^2 \pmod{p^{\alpha-2\beta}}} e\left(\frac{l}{p^{\alpha-\beta}}\right) \sum_{t \pmod{p^\beta}} e\left(\frac{t}{p^\beta}\right) \\
&= 0.
\end{aligned}$$

(ii) Suppose $c_b(m, p^\alpha) \neq 0$ where $(m, p) = 1$. We may assume $p^2 \nmid b$, for otherwise, $c_b(m, p^\alpha) = c_{b/p^2}(mp, p^\alpha) = 0$ by (i). Also $p \parallel b$ cannot happen because, when $\alpha \geq 2$, $p^2 \mid b$ if $p \mid b$ and $y^2 \equiv bm^2 \pmod{p^\alpha}$ has solutions. Thus $p \nmid b$.

Now $c_b(m, p^\alpha) \neq 0$ implies the congruence $y^2 \equiv bm^2 \pmod{p^\alpha}$ is soluble, and with $(m, p) = 1$, $y^2 \equiv b \pmod{p^\alpha}$ has two solutions, say, $\pm y_0$ and $p \nmid y_0$. We see that

$$\sum_{y^2 \equiv b \pmod{p^\alpha}} e\left(\frac{y}{p^\alpha}\right) = 2 \cos\left(2\pi \frac{y_0}{p^\alpha}\right) \neq 0$$

because otherwise, $y_0/p^\alpha = (2r+1)/4$ for some $r \in \mathbb{Z}$ or equivalently, $4y_0 = (2r+1)p^\alpha$ which contradicts to $p \nmid y_0$. \square

9. APPENDIX

Let us denote, as in [8, Section 3], the Kloosterman-Salié sum by

$$K_{2\ell+1}(m, n; c) := \sum_{d \pmod{c}} \varepsilon_d^{-(2\ell+1)} \left(\frac{c}{d}\right) e\left(\frac{md + n\bar{d}}{c}\right)$$

and

$$(9.1) \quad S(m, n; c) := \sum_{x \pmod{c}} \left(\frac{x}{c}\right) e\left(\frac{mx + n\bar{x}}{c}\right),$$

where $c \in \mathbb{N}$ and $m, n \in \mathbb{Z}$. Then we have the following estimate,

$$(9.2) \quad |K_{2\ell+1}(n, m; d)| \quad \text{and} \quad |S(m, n; d)| \leq d^{1/2} \tau(d) (d, n, m)^{1/2}$$

where $\tau(n)$ is the divisor function. This follows from the well-known Weil's bound for Kloosterman sums and the following lemma.

Lemma 9.1. *We have the following results:*

- (a) *Let $c = qr$ with $r \equiv 0 \pmod{4}$ and $(q, r) = 1$. Then*

$$K_{2\ell+1}(m, n; c) = K_{2\ell+2-q}(m\bar{q}, n\bar{q}; r) S(m\bar{r}, n\bar{r}; q)$$

where $q\bar{q} \equiv 1 \pmod{r}$ and $r\bar{r} \equiv 1 \pmod{q}$.

- (b) *Let q be odd, $q = uv$ with $(u, v) = 1$. Then*

$$S(m, n; q) = S(m\bar{u}, n\bar{u}; v) S(m\bar{v}, n\bar{v}; u)$$

where $u\bar{u} \equiv 1 \pmod{v}$ and $v\bar{v} \equiv 1 \pmod{u}$.

- (c) For an odd prime p and odd α , if $p \mid m$, then $S(m, 0; p^\alpha) = 0$.
- (d) If $(c, 2) = 1$, then $|S(m, n; c)| \leq (m, n, c)^{1/2} c^{1/2} \tau(c)$.
- (e) Let $4 \mid r \mid 2^\infty$. Then $|K_{2\ell+1}(m, n; r)| \leq (m, n, r)^{1/2} r^{1/2} \tau(r)$.

Proof. (a) See [8, p. 390, Lemma 2].

(b) See [8, p. 390, Lemma 3].

(c) By definition, for odd α , we have

$$S(m, 0; p^\alpha) = \sum_{x \pmod{p^\alpha}} \left(\frac{x}{p} \right) e\left(\frac{mx}{p^\alpha} \right).$$

When $\alpha = 1$, $S(m, 0; p^\alpha) = \sum_{x \pmod{p^\alpha}} \left(\frac{x}{p} \right) = 0$ as $p \mid m$. Suppose $\alpha \geq 3$. Putting $x = lp + v$, we get

$$\sum_{l \pmod{p^{\alpha-1}}} e\left(\frac{ml}{p^{\alpha-1}} \right) \sum_{v \pmod{p}} \left(\frac{v}{p} \right) e\left(\frac{mv}{p} \right) = 0.$$

(d) Iwaniec [9, Section 4.6] handled the case $(c, 2n) = 1$, and thus $(c, 2m) = 1$ too by symmetry. Together with (b), it suffice to deal with $p \mid (m, n)$ and c is a power of p .

Consider $S := S(p^a m, p^{a+b} n; p^{a+t})$ where $b \geq 0$, $p \nmid mn$, $a, t \geq 1$ and $a + t$ is odd. (The case that $a + t$ is even is done with the classical Kloosterman sum.) Clearly,

$$S = \sum_{d \pmod{p^{a+t}}} \left(\frac{d}{p} \right) e\left(\frac{md + p^b n \bar{d}}{p^t} \right) = \left(\frac{m}{p} \right) \sum_{d \pmod{p^{a+t}}} \left(\frac{d}{p} \right) e\left(\frac{d + p^b m n \bar{d}}{p^t} \right).$$

Mimicking Iwaniec's proof in [8, p. 67] (in fact attributed to Sarnak), we consider

$$F(x) = \sum_{d \pmod{p^{a+t}}} \left(\frac{d}{p} \right) e\left(\frac{x^2 d + p^b m n \bar{d}}{p^t} \right).$$

and its Fourier transform

$$\widehat{F}(y) = \sum_{x \pmod{p^t}} F(x) e\left(-\frac{xy}{p^t} \right).$$

As in [8, p. 67], we obtain $\widehat{F}(y) = g(1, p^t) G_t(4mnp^b - y^2)$ where

$$G_t(4mnp^b - y^2) = \sum_{d \pmod{p^{a+t}}} \left(\frac{d}{p} \right)^{t+1} e\left(\frac{d(4mnp^b - y^2)}{p^t} \right).$$

Case 1: t is odd. Then

$$\begin{aligned} G_t(4mnp^b - y^2) &= \sum_{d \pmod{p^{a+t}}}^* e\left(\frac{d(4mnp^b - y^2)}{p^t} \right) \\ &= \sum_{r=0,1} (-1)^r p^a \sum_{d \pmod{p^{t-r}}} e\left(\frac{d(4mnp^b - y^2)}{p^{t-r}} \right). \end{aligned}$$

Since

$$\sum_{d \pmod{p^{t-r}}} e\left(\frac{d(4mnp^b - y^2)}{p^{t-r}} \right) = p^{t-r} \delta_{y^2 \equiv 4mnp^b \pmod{p^{t-r}}},$$

we conclude

$$\widehat{F}(y) = g(1, p^t) \sum_{r=0,1} (-1)^r p^{a+t-r} \delta_{y^2 \equiv 4mnp^b \pmod{p^{t-r}}}$$

and

$$\begin{aligned} F(x) &= p^{-t} \sum_{y \pmod{p^t}} \widehat{F}(y) e\left(\frac{xy}{p^t}\right) \\ &= g(1, p^t) \sum_{r=0,1} (-1)^r p^{a-r} \sum_{\substack{y \pmod{p^t} \\ y^2 \equiv 4mnp^b \pmod{p^{t-r}}}} e\left(\frac{xy}{p^t}\right). \end{aligned}$$

As $|g(1, p^t)| \leq p^{t/2}$ by [9, (4.43)], we see that $|F(1)| \leq 2p^{a+t/2}$.

Case 2: t is even. Then

$$\begin{aligned} G_t(4mnp^b - y^2) &= \sum_{d \pmod{p^{a+t}}} \left(\frac{d}{p}\right) e\left(\frac{d(4mnp^b - y^2)}{p^t}\right) \\ &= \sum_{u \pmod{p^{a+t-1}}} e\left(\frac{u(4mnp^b - y^2)}{p^{t-1}}\right) \sum_{v \pmod{p}} \left(\frac{v}{p}\right) e\left(\frac{v(4mnp^b - y^2)}{p^{t-1}}\right). \end{aligned}$$

The first sum does not vanish only when $y^2 \equiv 4mn \pmod{p^{t-1}}$, but in this case, the second sum equals zero. i.e. $G_t(4mnp^b - y^2) = 0$. So $\widehat{F}(y) = g(1, p^t) G_t(4mnp^b - y^2) = 0$, implying $F(x) = 0$.

(e) Refer to [4], cf. [3, Section 14]. \square

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